If we combine (3), (4), and (5), statement (1) follows and equality is attained throughout if and only if $a = b = c$.

Solution 3 by Arkady Alt, San Jose, CA

Let $F = [ABC]$ (area) and let s be its semi-perimeter.
Since $h_a = \dfrac{2F}{a}, h_b = \dfrac{2F}{b}, h_c = \dfrac{2F}{c}$ and $abc = 4RF$ then

$$\begin{aligned}\sqrt[3]{\frac{1}{h_a h_b h_c}} &= \sqrt[3]{\frac{abc}{8F^3}} = \frac{1}{2F}\sqrt[3]{abc} \text{ and} \\ \frac{3abc}{2R}\sqrt[3]{\frac{1}{h_a h_b h_c}} &= 3\sqrt[3]{abc}.\end{aligned}$$

Thus, original inequality becomes

(1) $\quad \dfrac{a^2+bc}{b+c} + \dfrac{b^2+ca}{c+a} + \dfrac{c^2+ab}{a+b} \geq 3\sqrt[3]{abc}.$

Since $\dfrac{4a^2}{b+c} \geq 4a - b - c \iff (2a-b-c)^2 \geq 0$ we have

$$\begin{aligned}\sum_{cyc}\frac{a^2+bc}{b+c} &= \sum_{cyc}\frac{a^2}{b+c} + \sum_{cyc}\frac{bc}{b+c} \geq \sum_{cyc}\frac{4a-b-c}{4} + \sum_{cyc}\frac{bc}{b+c} \\ &= \frac{a+b+c}{2} + \sum_{cyc}\frac{bc}{b+c} = \sum_{cyc}\left(\frac{b+c}{4} + \frac{bc}{b+c}\right) \geq \sum_{cyc} 2\sqrt{\frac{b+c}{4}\cdot\frac{bc}{b+c}} \\ &= \sum_{cyc}\sqrt{bc} \geq 3\sqrt[3]{\sqrt{bc}\cdot\sqrt{ca}\cdot\sqrt{ab}} = 3\sqrt[3]{abc}.\end{aligned}$$

Solution 4 by Nicusor Zlota,"Traian Vuia" Technical College, Focsani, Romania, and Corneliu-Manescu Avram, Ploiesti, Romania

Assume that $a \geq b \geq c$.

First, we will prove that $\dfrac{a^2+bc}{b+c} + \dfrac{b^2+ca}{c+a} + \dfrac{c^2+ab}{a+b} \geq a+b+c \iff$

$\dfrac{a^2+bc}{b+c} - a + \dfrac{b^2+ca}{c+a} - b + \dfrac{c^2+ab}{a+b} - c \geq 0 \iff$

$\dfrac{(a-b)(a-c)}{b+c} + \dfrac{(b-c)(b-a)}{c+a} + \dfrac{(c-a)(c-b)}{a+b} \geq 0 \iff$

$(a-b)\left(\dfrac{a-c}{b+c} - \dfrac{b-c}{c+a}\right) + (b-a)\left(\dfrac{b-a}{c+a} - \dfrac{c-a}{a+b}\right) + (c-a)\left(\dfrac{a-c}{b+c} - \dfrac{b-c}{c+a}\right) \geq 0$

$(a-b)^2\dfrac{a+b}{(b+c)(c+a)} + (b-c)^2\dfrac{b+c}{(a+b)(c+a)} + (c-a)^2\dfrac{c+a}{(a+b)(b+c)} \geq 0.$